and help fix the principles according to which it works. Let us take one of the equations used in calibrating the balance. The equation[1]

$$x^3 + 3x^2 - 2x - 5 = 0$$

has a root between 1 and 2. Correct to 3 decimal places it is 1.330. This is outside the range of the balance. Form a new equation whose roots are half those of the original. The transformed equation is

$$8x^3 + 12x^2 - 4x - 5 = 0.$$

The root now sought is 0.665. Place the balance in equilibrium with buckets empty. The buckets on the curves $y = \pm x^4$ are left empty. In the bucket on $y = x^3$ (right) are placed 8 of the ball bearings. Similarly $y = x^2$ (right) is loaded with 12. The line $y = -x$ (left) is loaded with 4, and 5 ball bearings are placed in the bucket at the extremity of the left arm of the balance. Equilibrium is again established by means of the guide which is used to move the stirrups supporting the buckets to their proper position. The cross-hair should then be over 0.665 on the scale, the third figure being estimated. Obviously, the root of the original equation is 1.330.

The balance has stood the test of the class-room for a year and can be depended upon for roots correct to two decimal places, with a close approximation to the third. But more than for its worth in solving equations is it to be valued as an aid in fixing in the students' minds quite a number of the fundamental principles in the theory of equations.

GROUPS OF THE FIGURES OF ELEMENTARY GEOMETRY.

By G. A. MILLER, University of Illinois.

The groups of movements of figures of elementary geometry were considered in an article published in this MONTHLY, volume 10 (1903), page 214. The present note is complementary to this article, but the considerations are based upon entirely different principles. In fact, in the present note we inquire into the substitution group, on the sides or the edges of a figure, which transform this figure into others having the same absolute area or the same absolute volume; while the earlier considerations related to those movements which transform the figure as a whole into itself, but interchange some of its parts.

Let P represent any convex plane polygon of n sides. By interchanging two adjacent sides of P without affecting the other sides we can obtain another convex polygon with the same n sides and with the same absolute area. By a succession of such interchanges it is clearly possible to effect a transposition of any two sides without affecting the absolute area. Since every substitution on the n sides is the product of transpositions, it results that the $n!$ polygons obtained from P in the given manner have the same absolute area, and are conjugate under the symmetric group on their sides.

[1] Fite, *College Algebra*, pp. 192–7.

A necessary and sufficient condition that one of these polygons has a maximal area is that all of them have a maximal area, and hence they can all be inscribed in the same circle, as is also otherwise evident. In fact, this is a direct consequence of the well-known theorem that the circle incloses a larger area than any other plane figure having the same perimeter, since it is possible to construct a convex polygon inscribed in a circle whose sides are equal, in order, to those of any given convex polygon. As a special case there results the elementary theorem that we may interchange, according to any substitution of the symmetric group of degree n, the n sides of any convex polygon inscribed in a circle without affecting the area of this polygon.

The tetrahedron having no two equal edges and no edge which is equal to the sum of two of its edges presents an interesting elementary example from the present point of view. It is evident that the three edges of the base may be interchanged cyclically provided the other three edges are interchanged in the same order. That is, the absolute volume of the tetrahedron will not be affected by eight substitutions of the form $abc \cdot def$, since any one of the four faces may be used as a base. Moreover, if all the edges of a face are left invariant every interchange of the other three edges will in general affect the volume. It will be assumed that the tetrahedron in question has this property.

Hence the substitution group, composed of all the substitutions on the six edges which transform this tetrahedron into another having the same absolute volume, cannot involve more than 24 different substitutions since no more than six such substitutions are possible on the edges of a face. It is very easy to see that there are actually 24 such substitutions. In fact, every face can evidently be transformed into the base, and the edges of the base can be transformed in six different ways without affecting the absolute volume.

In addition to the eight substitutions of the given form there are nine of the form $ab \cdot cd$, and six of the form $abcd \cdot ef$, in addition to the identity. Hence it is easy to identify the substitution group obtained in the given manner as the transitive group of order 24 and of degree 6 which is composed of positive substitutions. This group was denoted by Cayley by the symbol $(+\ abcdef)_{24}$, *Quarterly Journal of Mathematics*, volume 25 (1891), page 81. As is well known it is simply isomorphic with the symmetric group of degree 4, and hence its abstract properties may be directly deduced from those of the latter group.

As there are 720 substitutions on the six edges of the given tetrahedron, and as only 24 of these transform this tetrahedron into one having the same absolute volume, it is generally possible to construct 30 tetrahedrons, having the given six edges, such that no two have the same absolute volume. This fact can also be seen by observing that 20 different triangles can be constructed with these six lines as sides. If we take any one of these triangles as a base it is possible to arrange the other three lines in such a way as to obtain six different tetrahedrons. Hence there result 120 tetrahedrons which have equal absolute volumes, in sets of four, since each face can be used as a base. For greater details along this line the reader may consult an article by Karl Schwering, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, volume 43 (1912), page 409.

ON A SPECIAL CASE OF THE TETRAHEDRAL COMPLEX.

By D. N. LEHMER, University of California

The teacher of line geometry often stands in need of simple examples to illustrate the general theory. The quadratic complex considered in this paper is a special, but interesting, case of the tetrahedral complex, and while the results might be obtained from the general theory the methods here given are worth noting for their simplicity.

For convenience of expression we shall say that a line in space is at right angles to a flat pencil of rays if it is at right angles to that ray of the pencil which it intersects. Using this definition we may state the theorem:

THEOREM. *The system of lines in space at right angles to a given flat pencil forms a quadratic complex of rays; that is, every point in space is the vertex of a quadratic cone the generating lines of which are lines of the system, and every plane in space contains a conic section the tangent lines of which are lines of the system.*

Let A (Fig. 1) be the center of the flat pencil of rays, and let the plane of the pencil be α. Let also a be any ray of A, and let N be the foot of the perpendicular

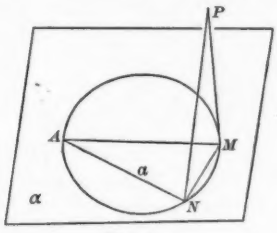

Fig. 1.

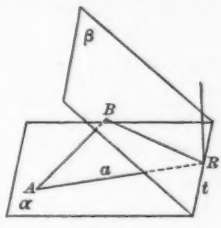

Fig. 2.

from any point P in space upon the line a. Also let M be the foot of the perpendicular from P upon the plane α. Then from elementary geometry ANM is a right angle, and the locus of N is a circle on AM as a diameter. Therefore the rays of the system through P meet the plane α in a circle, and are therefore the generating lines of a quadratic cone.

Further let β (Fig. 2) be any plane in space, and let B be the foot of the perpendicular from A to β. Let t be the line of intersection of α and β, and let R be the point of intersection of any ray a of the pencil A with the line t. Then that line of β which is at right angles to BR and which passes through R is a ray of the system. It follows from the elementary properties of the conic sections that the lines of the system lying in β are tangent to a parabola of which B is the focus, and t the tangent at the vertex. In fact, the lines are seen to be the rays joining corresponding points of two projective point-rows, t and the line at infinity.

Special points of a quadratic complex are such that the cones of rays at these points degenerate into pairs of planes. In the complex described above it is easily seen that the points in α are special. For if P is any point in α then any

line through P in the plane α will be at right angles to some line of A and therefore a line of the system. Also any line through P in a plane at right angles to the line AP will be a line of the system. The cone with vertex P has thus degenerated into a pair of planes. Similarly all the points in the plane at infinity are special. For let P be such a point and a that line of A which is at right angles to PA. Then any line through P in the plane of a and PA belongs to the system. Also any line through P in the plane at infinity belongs to the system. For consider any ray a' of A, and let Q be any point on it. Let a plane at right angles to a' at Q meet PA at P'. Then $P'Q$ is a ray of the system, and if Q moves to infinity P' also moves to infinity in the direction of P. The quadratic cone for any point at infinity is thus seen to degenerate into a pair of planes.

In *Special Planes* of the complex the conic touched by the lines of the system degenerates into a pair of points. This is seen to be the case for any plane through A, and also for any plane through the point at infinity in a direction normal to the plane α. These two points differ from other singular points in that *any* line through them is a line of the system.

Several interesting theorems are easily derived from the above discussion. Thus: *Given two flat pencils in different planes, it is possible to find at most four lines passing through any point of space and at right angles to both pencils.* For the two cones determined by the pencils at any point can have at most four elements in common. An unimportant exception arises when the pencils lie in parallel planes, and the point is chosen on the lines joining their centers. *Given two flat pencils in different planes, it is possible to find at most three finite lines lying in any plane and at right angles to both pencils.* For the parabolas determined by the two complexes in any plane can have at most three tangents in common. Taking account of the infinitely distant elements, however, we may state this and the preceding theorem: *The system of lines at right angles to two flat pencils which lie in different planes is of the fourth order and of the fourth class.*

As a final exercise for the student it is not difficult to prove: *The lines at right angles to two given flat pencils and which meet a given line in space are generators of a ruled quartic surface.*

ON THE USE OF PARTIAL DERIVATIVES IN PLOTTING CURVES FROM THEIR EQUATIONS.

By A. M. KENYON, Purdue University.

The article by M. O. Trip in the January, 1914, number of the MONTHLY on "An Application of Partial Derivatives to the Ellipse," has suggested that some further applications which have proved to be practically effective in tracing curves of the second and third degree may be of interest.

If $f(x, y) = 0$ is the equation (rational and integral in x and y) of a conic section, then $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ are equations of diameters which bisect all chords parallel to the x- and y-axes respectively, and cut the curve at points where

the tangents are horizontal or vertical. If x be eliminated between the equations $f = 0$ and $\partial f/\partial x = 0$ and y between $f = 0$ and $\partial f/\partial y = 0$, the resulting equations determine the horizontal and vertical tangents respectively.

In a central conic all diameters meet in the center; in a parabola all diameters are parallel to the axis of the curve and any two perpendicular ~~diameters~~ tangents meet on the directrix.

Let $\lambda : \mu$ denote the direction of a straight line which makes with the positive x-axis and with the positive y-axis, the angles whose cosines are respectively $\lambda/\sqrt{\lambda^2 + \mu^2}$ and $\mu/\sqrt{\lambda^2 + \mu^2}$. The direction of the line $ax + by + c = 0$ is $-b : a$ or $b : -a$.

If $f = 0$ is a conic, $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is a diameter which bisects all chords having the direction $\lambda : \mu$; if the direction $l : m$ of this diameter be determined (in terms of λ and μ), then the equation $\lambda l + \mu m = 0$ determines the directions of the principal axes of the conic; if it is a hyperbola the equation $\mu l - \lambda m = 0$ determines the directions of its asymptotes; if it is a parabola and if $p : q$ is the direction of $\partial f/\partial x = 0$ or $\partial f/\partial y = 0$, then $q(\partial f/\partial x) - p(\partial f/\partial y) = 0$ is the equation of the axis of the curve.

If $f = 0$ is a pair of parallel straight lines, then $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ represent a line midway between them; if it is a pair of intersecting lines then the equation $\mu l - \lambda m = 0$ determines their directions.

If $f(x, y) = 0$ is the equation (rational and integral in x and y) of a cubic such that no line in the direction $\lambda : \mu$ cuts it in three points, then $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is a conic (parabola, hyperbola, or pair of straight lines) which bisects all chords of the cubic which have this direction. In case this conic is a hyperbola one of its asymptotes has the direction $\lambda : \mu$ and is an asymptote of the cubic; if the conic is a pair of intersecting lines, one of them has the direction $\lambda : \mu$ and the other bisects all chords of the cubic having this direction; if the conic is a pair of parallel lines, both are asymptotes of the cubic.

EXAMPLES. (1) $2x^2 - 2xy + y^2 - 4x + 2y + 1 = 0$.

$\partial f/\partial x = 2(2x - y - 2)$ and $\partial f/\partial y = -2(x - y - 1)$ cross at the center. Elimination of x between $f = 0$ and $\partial f/\partial x = 0$ gives $y = \sqrt{2}$ and $y = -\sqrt{2}$, horizontal tangents; elimination of y between $f = 0$ and $\partial f/\partial y = 0$ gives $x = 0$ and $x = 2$, vertical tangents. $\partial f/\partial x + \partial f/\partial y = 0$ gives $x = 1$, a diameter bisecting all chords in the direction $1 : 1$; $\partial f/\partial x - \partial f/\partial y = 0$ gives $3x - 2y = 3$, which bisects chords in the direction $1 : -1$.

The direction of $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is $\lambda - \mu : 2\lambda - \mu$ and the equation $\lambda(\lambda - \mu) + \mu(2\lambda - \mu) = 0$ gives $\lambda : \mu = \sqrt{5} - 1 : 2$ and $\sqrt{5} + 1 : -2$, the directions of the principal axes. By drawing chords across these various diameters from the four points of tangency and other points easily found on the curve, a large number of new points may be located.

(2) $$x^2 - 4xy + y^2 + 6x + 24 = 0.$$

$\partial f/\partial x = 2(x - 2y + 3)$ and $\partial f/\partial y = 2(y - 2x)$ cross at the center. Elimination of x between $f = 0$ and $\partial f/\partial x = 0$ gives $y = -1$, $y = 5$, horizontal tangents;

similarly $x=-2, x=4$ are vertical tangents. The equation $\lambda(2\lambda-\mu)+\mu(\lambda-2\mu)=0$ gives $\lambda:\mu = 1:1$ and $1:-1$, the directions of the principal axes; the equation $\mu(2\lambda-\mu)-\lambda(\lambda-2\mu)=0$ gives $\lambda:\mu = 2+\sqrt{3}:1$ and $2-\sqrt{3}:1$, the directions of the asymptotes.

(3) $$x^2+2xy+y^2-2x-6y+1=0.$$

$\partial f/\partial x = 2(x+y-1)$ and $\partial f/\partial y = 2(x+y-3)$ have the same direction $1:-1$. $-\partial f/\partial x-\partial f/\partial y=0$ gives $x+y=2$, the axis of the curve. Elimination of x between $f=0$ and $\partial f/\partial x=0$ gives $y=0$, the horizontal tangent. Similarly $x=2$ is the vertical tangent; these tangents cross on the directrix. The focus may now be located.

(4) $$x^3+2y^3-3xy^2-6x-6y=0.$$

$\partial f/\partial x = 3(x^2-y^2-2)$ and $\partial f/\partial y = 6(y^2-xy-1)$. Lines in the two directions $1:1$ and $2:-1$ cut the cubic in two points only. $\partial f/\partial x+\partial f/\partial y=0$ gives $(x-y+2)(x-y-2)=0$, both asymptotes of the cubic; $2(\partial f/\partial x)-\partial f/\partial y=0$ gives $x^2+xy-2y^2=1$, a hyperbola whose center is at the origin and whose asymptotes are $x-y=0$ and $x+2y=0$, the latter being also an asymptote of the cubic. This hyperbola bisects all chords having the direction $2:-1$.

A METHOD OF SOLVING NUMERICAL EQUATIONS.

By S. A. COREY, Hiteman, Iowa.

The following development of the roots of an equation by Maclaurin's formula applies to both algebraic and transcendental equations, and gives all the roots approximately whenever the conditions involved in the development can be fully complied with.

Let $f(r)=0$ be an equation to be solved, and let a be an approximation to a root r of the equation.

Let us suppose that $f(r)$ is single-valued and analytic in a circle about a as a center and including r. Then

$$f(r)=f(a)+f'(a)(r-a)+\frac{f''(a)}{2!}(r-a)^2+\frac{f'''(a)}{3!}(r-a)^3+\cdots.$$

Hence, by the usual formulas for the reversion of series,[1] putting $z=f(r)-f(a)$, we get

(1)
$$r-a=\left(\frac{dr}{dz}\right)_0 z+\left(\frac{d^2r}{dz^2}\right)_0\frac{z^2}{2!}+\left(\frac{d^3r}{dz^3}\right)_0\frac{z^3}{3!}+\cdots\left(\frac{d^nr}{dz^n}\right)_0\frac{z^n}{n!}+\cdots$$
$$=A_1^{-1}z-A_2A_1^{-3}\frac{z^2}{2!}+(3A_2^2A_1^{-5}-A_3A_1^{-4})\frac{z^3}{3!}+\cdots A_1^{-1}\frac{d}{da}\left(\frac{dr^{n-1}}{dz^{n-1}}\right)_0\frac{z^n}{n!}+\cdots,$$

where $A_k=d^kf(a)/da^k$ and $A=f(a)$. In practice it is necessary that the devel-

[1] See, for instance, Goursat, *A Course in Mathematical Analysis*, Vol. 1, §§ 189–190 (first edition).

opment in (1) should be rapidly convergent for the number of terms employed, and the conditions of convergence for the development make it possible to determine the degree of accuracy attained by using a given number of terms.

It will be observed that of all the derivatives A_k in (1), A_1 is the only one which has or can have a negative exponent, and hence the only one which must never have a zero modulus for any value of a as it approaches r. It will also be observed that in order that (1) may be rapidly convergent it is essential that modulus z should be as small as possible, and that modulus A_1 should be as large as possible.

Employing only the terms of (1) preceding the term containing z^4 and putting $z = f(r) - f(a) = -A$, since $f(r) = 0$ for a root r and $A = f(a)$, we get the practical working formula:

$$r = a - AA_1^{-1} - \tfrac{1}{2}A^2A_2A_1^{-3} - \tfrac{1}{2}A^3A_2^2A_1^{-5} + \tfrac{1}{6}A^3A_3A_1^{-4} \cdots, \qquad (2)$$

which is quite well adapted to logarithmic computation for both real and imaginary values of the roots.

Should the degree of accuracy attained by using (2) not be sufficiently great when the first assumed value of a is employed, greater accuracy may be obtained by substituting the new value, and so on, until the required degree of accuracy is attained.

To more clearly indicate the use of the method the following examples will be useful.

Let $f(x) = x^3 + 2x - \sin x - 15 = 0$ be an equation, a real positive root of which, accurate to five decimal places, is to be found.

We know from Sturm's theorem, or can learn by trial, graphic methods or otherwise, that a real root lies between 2 and 3. Therefore assuming that $a = 2$, we get,

$$A = 2^3 + 2 \cdot 2 - \sin 2 - 15 = - \ 3.9093$$

$$A_1 = 3 \cdot 2^2 + 2 - \cos 2 \quad = + \ 14.4161$$

$$A_2 = 3 \cdot 2 \cdot 2 + \sin 2 \quad = + \ 12.9093$$

$$A_3 = 3 \cdot 2 + \cos 2 \quad = + \ 5.5839$$

Substituting in (2), we get $x = 2.244$. Again letting $a = 2.244$, finding new values of A, A_1, A_2, A_3, and substituting in (2), we get $x = 2.243666$, the accuracy of which is determined not by the convergence of (2) but by the values of the sine and cosine as given in the ordinary 6-place tables.

As a second example let us take the quartic equation

$$f(x) = x^4 - 3x^2 + 75x - 10{,}000 = 0,$$

which Merriman gives on page 34 of his *Solution of Equations*, 4th ed., to illustrate Lambert's method. Then taking, as he does, the approximate values of the roots to be the four roots of unity each multiplied by 10, we get

				Modulus	angle
$a=+\ 10$	$-\ 10$	$+10i$	$-10i$		
$A=+\ 450$	$-1{,}050$	$+\ 300+\ 750i$	$+\ 300-\ 750i$	$\sqrt{652{,}500}$	$68°\ 11'\ 55''$
$A_1=+4{,}015$	$-3{,}865$	$+\ 75-4{,}060i$	$+\ 75+4{,}060i$	$\sqrt{16{,}489{,}225}$	$-(88°\ 56'\ 30'')$
$A_2=+1{,}194$	$+\ 1{,}194$	$-1{,}206$	$-1{,}206$	$1{,}206$	$180°\ 0'\ 0''$
$A_3=+\ 240$	$-\ 240$	$+\ 240i$	$-\ 240i$	240	$90°\ 0'\ 0''$
$A_4=+\ 24$	$+\ 24$	$+\ 24$	$+\ 24$	24	$0°\ 0'\ 0''$

Substituting in the terms of (2)

$a=$	$+10.00000$	-10.00000	$+10i$	$-10i$
$-AA_1^{-1}=$	$-\ 0.11208$	$-\ 0.27167$	$+0.18330-0.07728i$	$+0.18330+0.07728i$
$-\frac{1}{2}A^2A_2A_1^{-3}=$	$-\ 0.00186$	$+\ 0.01140$	$+0.00428+0.00402i$	$+0.00428-0.00402i$
$-\frac{1}{2}A^2A_2^2A_1^{-5}=$	$-\ 0.00006$	$-\ 0.00095$	$-0.00012-0.00033i$	$-0.00012+0.00033i$
$+\frac{1}{6}A^3A_3A_1^{-4}=$	$+\ 0.00001$	$+\ 0.00021$	$0.00000 \cdot 0.00000$	$0.00000\ \ 0.00000$
$x=\text{sum}\ =$	$+\ 9.88601$	-10.26101	$+0.18746+9.92641i$	$+0.18746-9.92641i$

Should still greater accuracy be required replace the above values of a by the values of x just found, and so on until the required degree of accuracy is attained.

A FORMULA FOR THE SUM OF A CERTAIN TYPE OF INFINITE POWER SERIES.[1]

By ELBERT H. CLARKE, Purdue University.

INTRODUCTION.

The problem to be considered is that of finding a definite, finite formula which will give the sum of any convergent infinite series whose terms are such that their numerators form an arithmetical progression of any order[2] and whose denominators form a geometric progression.

Since the nth term of an arithmetic progression of the kth order may be reduced to the form

$$a_n = b_0n^k + b_1n^{k-1} + \cdots + b_k,$$

our problem is to evaluate the expression

$$T = \sum_{n=1}^{\infty} \frac{b_0n^k + b_1n^{k-1} + \cdots + b_k}{ar^n}$$

in which the b's are independent of n. But this may be written

$$T = \frac{b_0}{a}\sum_{n=1}^{\infty}\frac{n^k}{r^n} + \frac{b_1}{a}\sum_{n=1}^{\infty}\frac{n^{k-1}}{r^n} + \cdots + \frac{b_k}{a}\sum_{n=1}^{\infty}\frac{1}{r^n}$$

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[1] The author wishes to acknowledge criticisms and suggestions from Professors R. D. Carmichael and A. C. Lunn.

[2] See *Text Book of Algebra*, CHRYSTALL, Vol. 1, page 484.

and so the problem is at once reduced to that of finding a formula for

$$\sum_{n=1}^{\infty} \frac{n^k}{r^n} \quad (|r| > 1,\ k \text{ a positive integer or zero})$$

which will be denoted in what follows by $S_{k,\,r}$.

The Form of the Sum.

The series $S_{k,\,r}$, being a power series in $1/r$, may be differentiated term by term with respect to r in its region of convergence.

$$\frac{dS_{k,\,r}}{dr} = \sum_{n=1}^{\infty} -\frac{n^{k+1}}{r^{n+1}} = -\frac{1}{r}\sum_{n=1}^{\infty} \frac{n^{k+1}}{r^n}.$$

This gives at once a fundamental relation,

$$S_{k+1,\,r} = -\,r\,\frac{dS_{k,\,r}}{dr}. \tag{1}$$

Furthermore, $S_{0,\,r}$ is none other than the ordinary geometric series, so we have at once

$$S_{0,\,r} = \frac{1}{r-1}$$

and we can derive any particular $S_{k,\,r}$ by k applications of the fundamental relation (1).

By an inspection of the forms of $S_{1,\,r}$; $S_{2,\,r}\cdots$; we are led to expect

$$S_{k,\,r} = \frac{r}{(r-1)^{k+1}} F_{k-1}(r),$$

where $F_{k-1}(r)$ is a polynomial of degree $(k-1)$ in r. Applying (1) we obtain, after a few simple reductions,

$$S_{k+1,\,r} = \frac{r}{(r-1)^{k+2}}\left[(kr+1)F_{k-1}(r) + (r-r^2)F'_{k-1}(r)\right]. \tag{2}$$

If $F_{k-1}(r)$ is a polynomial of degree $(k-1)$ in r then the bracketed quantity is a polynomial of degree k in r and may be called $F_k(r)$. In this way the form of $S_{k,\,r}$ is determined except for the particular coefficients occurring in the polynomial $F_{k-1}(r)$.

Determination of the Coefficients.

Let us write

$$F_{k-1}(r) = \alpha_{k-1,\,1}\,r^{k-1} + \alpha_{k-1,\,2}\,r^{k-2} + \alpha_{k-1,\,3}\,r^{k-3} + \cdots + \alpha_{k-1,\,k-1}\,r + \alpha_{k-1,\,k}.$$

Then by carrying out the operations indicated inside the brackets in (2) we obtain

$$\begin{aligned} F_k(r) = \alpha_{k-1,\,1}\,r^k &+ [2\alpha_{k-1,\,2} + k\alpha_{k-1,\,1}]r^{k-1} \\ &+ \cdots + [t\alpha_{k-1,\,t} + (k-t+2)\alpha_{k-1,\,t-1}]r^{k-t+1} \\ &+ \cdots + [k\alpha_{k-1,\,2} + 2\alpha_{k-1,\,k-1}]r + \alpha_{k-1,\,k}. \end{aligned} \tag{3}$$

The following relation is seen to exist between the coefficients of $F_k(r)$ and $F_{k-1}(r)$:

$$\alpha_{k,\,t} = t\alpha_{k-1,\,t} + (k - t + 2)\alpha_{k-1,\,t-1}.$$

By inspection of $F_1(r)$, $F_2(r)$ $\cdots$ we notice that the coefficients are symmetrically arranged and that the first and last coefficients are always 1. By our work we see that the first and last coefficients in $F_{k-1}(r)$ and $F_k(r)$ are the same, therefore they must be 1. Let us assume further a symmetrical arrangement of the coefficients in $F_{k-1}(r)$. That is to say $\alpha_{k-1,\,t} = \alpha_{k-1,\,k-t+1}$. This assumption, in connection with the above recurrence relation between the coefficients of $F_k(r)$ and $F_{k-1}(r)$, gives us immediately

$$\alpha_{k,\,t} = t\alpha_{k-1,\,t} + (k - t + 2)\alpha_{k-1,\,t-1} = (k - t + 2)\alpha_{k-1,\,k-t+2} + t\alpha_{k-1,\,k-t+1} = \alpha_{k,\,k-t+2}. \tag{4}$$

So that symmetry of arrangement of the coefficients in $F_{k-1}(r)$ implies symmetry of arrangement of the coefficients in $F_k(r)$. That the coefficients in $F_k(r)$ are symmetrically arranged follows at once from an inspection of a particular $F_k(r)$, e. g., $F_4(r) = r^4 + 26r^3 + 66r^2 + 26r + 1$.

It is an interesting fact that the sum of the coefficients in $F_k(r)$ is $(k + 1)!$. The reader may easily derive the proof from (3).

As already stated, the *first* coefficient satisfies the relation

$$\alpha_{k,\,1} = \alpha_{k-1,\,1} = \cdots = 1.$$

By carrying out the recurrence relations for the *second* coefficient we obtain

$$\alpha_{2,\,2} = 2\alpha_{2,\,1} + 2, \qquad (\alpha_{2,\,1} = 1)$$

$$\alpha_{3,\,2} = 2^2 \cdot 1 + 2 \cdot 2 + 3,$$

$$\alpha_{4,\,2} = 2^3 \cdot 1 + 2^2 \cdot 2 + 2 \cdot 3 + 4,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{k,\,2} = 2^{k-1} \cdot 1 + 2^{k-2} \cdot 2 + 2^{k-3} \cdot 3 + \cdots + 2^2(k - 2) + 2(k - 1) + k.$$

If we write $\alpha_{k,\,2}$ in rows and add by columns we shall obtain without difficulty

$$\alpha_{k,\,2} = 2^{k+1} - (k + 2).$$

To determine the *third* coefficient we may proceed as above making use of the following formulas:

$$\sum_{k=0}^{e} 3^{e-k} \cdot 2^k = 3^{e+1} - 2^{e+1},$$

$$\sum_{k=0}^{e} 3^{e-k} \cdot k^2 = \sum_{k=0}^{e} 3^{e-k} \cdot k + \tfrac{1}{2}\sum_{k=0}^{e} 3^{e-k} - \tfrac{1}{2}(e + 1)^2,$$

$$\sum_{k=0}^{e} 3^{e-k} \cdot k = \tfrac{1}{2}\sum_{k=0}^{e} 3^{e-k} - \tfrac{1}{2}(e + 1).$$

When we have gone through with the necessary work, which is not difficult but is tedious, we obtain the following result

$$\alpha_{k,3} = 2^2[3^{k-2}\cdot 1 + 3^{k-3}\cdot 2\cdot 2 + \cdots + 3(k-2)2^{k-3} + (k-1)2^{k-2}]$$
$$-[3^{k-2}\cdot 1\cdot 3 + 3^{k-3}\cdot 2\cdot 4 + \cdots + 3(k-2)k + (k-1)(k+1)]$$
$$= 3^{k+1} - 2^{k+1}(k+2) + \frac{(k+2)(k+1)}{2}.$$

It is quite possible, of course, to proceed in the same way to obtain the fourth coefficient and so on. But the work will be extremely tedious and will not be likely to render much easier the generalization by inspection which must be made in any case. By an inspection of the results for $\alpha_{k,2}$ and for $\alpha_{k,3}$, together with (3) we are led to assume for the tth coefficient in $F_k(r)$,

$$\alpha_{k,t} = \left[t^{k+1} - (t-1)^{k+1}\binom{k+2}{1} + \cdots + (-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1}\right.$$
$$\left. + \cdots + (-1)^{t-1}\binom{k+2}{t-1}\right],$$

which gives the proper forms for $\alpha_{k,2}$ and $\alpha_{k,3}$ above and which further yields proper numerical values for $\alpha_{1,4}, \alpha_{2,4}, \cdots$. All that is needed to complete the induction is a proof that $\alpha_{k,t}$ will satisfy the recurrence relation (4). That is to say,

$$t^{k+1} - (t-1)^{k+1}\binom{k+2}{1} + \cdots + (-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1} + \cdots$$
$$+ (-1)^{t-2}2^{k+1}\binom{k+2}{t-2} + (-1)^{t-1}\binom{k+2}{t-1}$$

must be equal to

$$t\left[t^k - (t-1)^k\binom{k+1}{1} + \cdots + (-1)^{s-1}(t-s+1)^k\binom{k+1}{s-1} + \cdots\right.$$
$$\left. + (-1)^{t-1}\binom{k+1}{t-1}\right] + (k-t+2)\left[(t-1)^k - (t-2)^k\binom{k+1}{1} + \cdots\right.$$
$$\left. + (-1)^{s-1}(t-s)^k\binom{k+1}{s-1} + \cdots + (-1)^{t-2}\binom{k+1}{t-2}\right].$$

Carrying out the work, the latter expression becomes

$$t^{k+1} - (t-1)^k t\binom{k+1}{1} + \cdots + (1)^{s-1}(t-s+1)^k t\binom{k+1}{s-1} + \cdots + (-1)^{t-1}t\binom{k+1}{t-1}$$
$$+ (t-1)^k(k-t+2) + \cdots + (-1)^{s-2}(t-s+1)^k(k-t+2)\binom{k+1}{s-2} + \cdots$$
$$+ (-1)^{t-2}(k-t+2)\binom{k+1}{t-2}.$$

Combining like terms, we have

$$t^{k+1}-(t-1)^k[t(k+1)-k+t-2]+\cdots+(-1)^{s-1}(t-s+1)^k\binom{k+1}{s-2}$$
$$\times\left[t\frac{k-s+3}{s-1}-k+t-2\right]+\cdots+(-1)^{t-1}\binom{k+1}{t-2}\left[t\frac{k-t+3}{t-1}-k+t-2\right],$$

which reduces to the left member above:

$$t^{k+1}-(t-1)^{k+1}\binom{k+2}{1}+\cdots+(-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1}+\cdots$$
$$+(-1)^{t-1}\binom{k+2}{t-1}.$$

If we replace k by $k-1$ we shall have the tth term in $F_{k-1}(r)$. Hence we have the desired formula for $S_{k,r}$:

$$S_{k,r}=\sum_{n=1}^{\infty}\frac{n^k}{r^n}=\frac{r}{(r-1)^{k+1}}\Big\{r^{k-1}+[2^k-(k+1)]r^{k-2}+\cdots$$
$$+\left[t^k+\cdots+(-1)^{s-1}(t-s+1)^k\binom{k+1}{s-1}+\cdots\right.$$
$$\left.+(-1)^{t-1}\binom{k+1}{t-1}\right]r^{k-t}+\cdots+1\Big\}.$$

The general series of the introductory paragraph may now be written

$$T=\frac{b_0}{a}S_{k,r}+\frac{b_1}{a}S_{k-1,r}+\cdots+\frac{b_r}{a}S_{0,r}.$$

BOOK REVIEWS.

Memorabilia Mathematica. By ROBERT EDOUARD MORITZ. vii+410 pages. The Macmillan Co., New York, 1914. $3.00 net.

In the Memorabilia Mathematica we are presented with a collection of more than 1100 quotations pertaining to many phases of mathematics and to the life and thought of mathematicians. The page preceding the title page is graced by two quotations from Goethe and Emerson. They read: "Alles Gescheite ist schon gedacht worden; man muss nur versuchen es noch einmal zu denken," and "A great man quotes bravely, and will not draw on his own invention when his memory serves him with a word as good."

We may safely infer that in these quotations, if not interpreted too literally, the author expresses a conviction and implies a purpose. The intrinsic value of the ideas expressed in the Memorabilia shows that the author's conviction is well founded. His purpose is realized in bringing this wealth of ideas within easy reach of mathematical and non-mathematical students.

The volume presents a great range of subjects. It deals not only with the nature and value of mathematics, with the relation of mathematics to logic, philosophy, and science, and with mathematics as a fine art and as a language; but it also introduces a strong personal, human element and deals with the mathematician. In a work of this kind we expect to find many mathematical authors quoted; but are hardly prepared to meet, in addition to the mathematical authors, the names of many men famous in other walks of life. But such is the case. We meet Benjamin Franklin and Daniel Webster, John Locke and Francis Bacon, Goethe and Froebel, Napoleon and Voltaire, and many others.

The book is very suggestive and stimulating. Many viewpoints that are presented contain a challenge to the thinking mind. In dealing with the nature of mathematics we thus find, from the pen of Klein, "Mathematics is fundamentally the science of self evident things," to which Pringsheim replied: "For the majority of the cultured, even of scientists, mathematics remains the science of the incomprehensible." The remark of Huxley certainly contains a challenge: "Mathematics is that subject which knows nothing of experiment, nothing of induction, nothing of causation."

In the light of the fact that mathematics is often spoken of as the science that deals largely with logical processes, and that draws necessary conclusions, the following from Voltaire is significant: "There is an astonishing imagination, even in the science of mathematics. . . . We repeat, there was far more imagination in the head of Archimedes than in that of Homer." From another writer, bearing on the same subject and emphasizing other qualities than that of logic, comes the dictum: "The whole of mathematics consists of a series of aids to the imagination in the process of reasoning." Keyser introduces us to a significant, interesting encounter: "When the greatest of American logicians, speaking of the powers that constitute the born geometer, had mentioned conception, imagination, and generalization, he paused. Thereupon from the audience there came the challenge, 'What of reason?' The instant response, not less just than brilliant, was: 'Ratiocination—that is but the smooth pavement on which the chariot rolls.'"

The Memorabilia has rendered a real service in presenting quotations of sufficient length to illuminate and give a setting for ideas, which, without such a setting, are incomprehensible and misleading. Thus the quotation of Russell, at times quoted with startling effect, that "Mathematics may be defined as the subject in which we never know what we are talking about, or whether what we are saying is true," might be taken as that of a cynic or an iconoclast; but the Memorabilia presents this and other passages with enough of the context, so that they come to us as significant, constructive contributions.

The Memorabilia also presents us with many a human touch, and throws a light, not to be found in purely mathematical books, on many well known men. To take but one case, let us consider Newton for a moment. We find

> "Nature and nature's law lay hid in night:
> God said, 'Let Newton be!', and all was light."

and

"A monument to Newton! A monument to Shakespeare! Look up to heaven—look into the Human Heart. Till all the planets and the passions—the affections and the fixed stars are extinguished—their names cannot die."

In contrast to this, on the other hand, we are told that Newton "could not readily make up a common account: and, when master of the mint, used to get somebody else to make up his accounts for him." Another side light is the mention of his relation to astrology. "There was a time when Newton was possessed with the old fooleries of astrology; and another when he was so far gone in those of chemistry as to be upon the hunt after the philosopher's stone."

Of the modesty of this great man we are told "If I have seen farther than Descartes, it is by standing on the shoulders of giants." And again: "Newton could not admit that there was any difference between him and other men, except in the possession of such habits as . . . perseverance and vigilance. When he was asked how he made his discoveries, he answered, 'By always thinking about them'; and at another time he declared that if he had done anything, it was due to nothing but industry and patient thought: 'I keep the subject of my inquiry constantly before me, and wait till the first dawning opens gradually, by little and little, into a full and clear light.'"

The devotees in many ages have sounded the praises of mathematics. From Euripides the Memorabilia presents us with this contribution: "Mighty are numbers, joined with art resistless"; and from Plato comes a passage that contains a claim for arithmetic which has been heard for mathematics in general in more modern days, for we read: "No single instrument of youthful education has such mighty power, both as regards domestic economy and politics, and in the arts, as the study of arithmetic. Above all, arithmetic stirs up him who is by nature sleepy and dull, and makes him quick to learn, retentive, shrewd, and aided by art divine he makes progress quite beyond his natural powers." Another note of praise is sounded in the words: "If the Greeks had not cultivated Conic Sections, Kepler could not have superseded Ptolemy; if the Greeks had cultivated Dynamics, Kepler might have anticipated Newton."

The relation of mathematics to our every day experiences is expressed in such passages as the following: "Mathematics, the science of the ideal, becomes the means of investigating, understanding and making known the world of the real." And again: "Geometry in every proposition speaks a language which experience never dares to utter; and indeed of which she but half comprehends the meaning. Experience sees that the assertions are true, but she sees not how profound and absolute is their truth. She unhesitatingly assents to the laws which geometry delivers, but she does not pretend to see the origin of their obligation. She is always ready to acknowledge the sway of pure scientific principles as a matter of fact, but she does not dream of offering her opinion on their authority as a matter of right; still less can she justly claim to herself the source of that authority."

A delicious humor is found in many passages. I wish to give but a single

example from De Morgan. "German intellect is an excellent thing, but when a German product is presented it must be analyzed. Most probably it is a combination of intellect (I) and tobacco-smoke (T). Certainly I_3T_1, and I_2T_1 occur; but I_1T_3 is more common, and I_2T_{15} and I_1T_{20} occur. In many cases metaphysics (M) occurs and I hold that $I_aT_bM_c$ never occurs without $b+c>2a$.

N.B.—Be careful, in analyzing the compounds of the three, not to confound T and M, which are strongly suspected to be isomorphic. Thus, $I_1T_3M_3$ may easily be confounded with I_1T_6. As far as I dare say anything, those who have placed *Hegel*, *Fichte*, etc., in the rank of the extenders of *Kant* have imagined T and M to be identical."

I have quoted freely from the volume under review, hoping thereby to convey something of the atmosphere that pervades it; but I realize fully that no limited number of quotations can adequately suggest the wealth and variety of ideas that it presents to the reader.

The author was very happy in the selection of material in this pioneer work, and has rendered a service to mathematical and non-mathematical readers.

The value of the work is greatly enhanced by the author's success in giving exact references to the quotations used, for in this way the Memorabilia becomes a guide to a much larger range of material "pertaining to mathematics, by poets, philosophers, statesmen, scientists, and mathematicians."

An excellent cross reference index of some 700 topics makes the material gathered very accessible.

GEO. N. BAUER.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up solutions of 412 to 421 had been received. A solution of 406 is desired.

422. Proposed by W. D. CAIRNS, Oberlin College.

Find a solution of the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$.
(Adapted from Godfrey & Siddon's *Elementary Algebra*.)

423. Proposed by ELBERT H. CLARKE, Purdue University.

Show that the following formula is true for all positive integral values of k. The parenthetical symbols are defined as being the binomial coefficients. $\binom{k+1}{0} = 1$, by definition.

$$k^k\binom{k+1}{0} - (k-1)^k\binom{k+1}{1} + (k-2)^k\binom{k+1}{2} + \cdots + (-1)^{s-1}(k-s+1)^k\binom{k+1}{s-1} + \cdots + (-1)^k\binom{k+1}{k-1} = 1.$$

GEOMETRY.

When this issue was made up no solutions of 444, 446, 449 had been received. Please give attention to these.

451. Proposed by CLIFFORD N. MILLS, South Dakota State College.

Determine the sides of an isosceles triangle of given area, having given that the sum of its sides is equal to the sum of its base and altitude.

CALCULUS.

When this issue was made up no solutions of 348, 353, 354, 360, 363 had been received.

372. Proposed by V. M. SPUNAR, Chicago, Ill.

Find the condition that the equation:

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{a^2}{x^2}\right)y=0$$

should have one solution expressible in integral powers of x; and show that when this condition is satisfied, every other solution of the equation possesses a logarithmic infinity at the origin.

373. Proposed by C. N. SCHMALL, New York City, N. Y.

In the *Encyclopaedia Britannica* article on "Capillary Action" (Vol. 5, p. 268, 11th ed.) it is shown that $1/R_1 + 1/R_2 = p/T$, in the case of a soap bubble, where R_1, R_2 are the radii of curvature of a meridian section and a normal section, respectively, of the bubble; p, the difference of air-pressure; T, the energy per unit area of the film. Employing the principle that the soap bubble tends to assume a form such that the area of its surface is a *minimum* for a *given volume* of air, show by the Calculus of Variations that $1/R_1 + 1/R_2 = k$, where k is a constant.

MECHANICS.

Solutions of 286, 287, 288, 290, 291, 298, 299, 300 are desired.

300. Proposed by V. M. SPUNAR, Chicago, Ill.

A helical spring is composed of 20 turns of steel wire .258″ in diameter, the diameter of the coil being 3″. If the spring is compressed by a force of 50 lb., what is the maximum stress in the spring, its axial compression, and its resilience?

SOLUTIONS OF PROBLEMS.

ALGEBRA.

410. Proposed by C. N. SCHMALL, New York City.

Solve the simultaneous equations,

$$x^2 + xy + y^2 = a,$$
$$x^4 + x^2y^2 + y^4 = b.$$

SOLUTION BY HORACE OLSON, Chicago, Ill.

The equations may be written

$$x^2 + xy + y^2 = a, \tag{1}$$

$$(x^2 + xy + y^2)(x^2 - xy + y^2) = b. \tag{2}$$

Whence

$$x^2 - xy + y^2 = b/a. \tag{3}$$

From (1) and (3),

$$x^2 + y^2 = (a^2 + b)/2a \quad \text{and} \quad 2xy = (2a^2 - b)/2a.$$

From these equations

$$x + y = \pm\sqrt{(3a^2 - b)/2a} \qquad \text{and} \qquad x - y = \pm\sqrt{(3b - 2a)/2a}.$$

Whence

$$x = \frac{1}{4a}[\pm\sqrt{6a^3 - 2ab} \pm \sqrt{6ab - 2a^3}],$$

$$y = \frac{1}{4a}[\pm\sqrt{6a^3 - 2ab} \mp \sqrt{6ab - 2a^3}].$$

Solved similarly by G. W. HARTWELL, PAUL CAPRON, BERNARD KRAMER, W. C. EELLS, CLIFFORD N. MILLS, J. L. RILEY, G. R. MIRICK, H. C. FEEMSTER, C. E. GITHENS, A. A. NAUER, ELMER SCHUYLER, and the PROPOSER.

Editorial Note.

These solutions are all conventional and do not state clearly,

(1) Which x's go with which y's. The four solutions obtained are, as to signs of the radicals,

x	$++$	$+-$	$-+$	$--$
y	$+-$	$++$	$--$	$-+$

(2) Necessary and sufficient conditions for real roots.

They are

$$a > 0, \quad b > 0, \quad 9a^2 > 3b > a^2.$$

(3) The number of solutions is not that demanded by Bezout's theorem. There is no way of being sure that all the real solutions have been found except by producing the complete set of eight solutions and discarding those not acceptable. In this case the loci (1) and (2) have ordinary contact at each of the complex points at infinity in the directions whose slopes are ω and ω^2 (cube roots of unity). This accounts for the other four points and clinches the argument as to the real intersections.

GEOMETRY.

421. Proposed by R. P. BAKER, University of Iowa.

Assuming the details of the proof of the existence of a sphere inscribed in a tetrahedron as usual in the texts, give an intuitional proof that there are in general eight spheres each touching the four faces, but for the regular tetrahedron only five. How many special types are there?

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Call the planes passing through the six edges and bisecting the dihedral angles internally and externally respectively $a, a'; b, b'; c, c'; d, d'; e, e'; f, f'$. Call the planes BCD, CDA, DAB, and ABC I, II, III, and IV respectively (fig. 1).

A point will be said to be on the positive side of a face if it is on the same side of that face as a point inside the tetrahedron; if not it will be said to be on the negative side.

Now every point in a is equidistant from III and IV with same signs; every point in a' is equidistant from III and IV with opposite signs. So every point in b is equid stant from II and IV with same signs; every point n b' equidistant from II, IV with opposite signs. Similarly c and c' with respect to II and III;

d and d' with respect to I and IV; e and e' with respect to I and II; f and f' with respect to I and III.

It is easily proved, then, that a, b, c, d, e, f meet at O, a point equidistant from all faces and having the same sign with respect to all, i. e., the center of the inscribed sphere. Similarly that a, b, c, d', e', f' meet at E_7 a point equidistant from all four faces, the distances from II, III, IV having the same sign and that from I the opposite sign, i. e., the center of an escribed sphere touching the face I on the opposite side to the inscribed sphere. Similarly a, d, f, b', c', e' determine

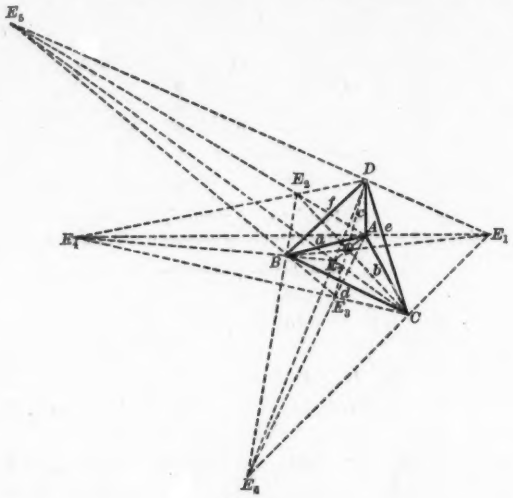

FIG. 1.

E_1, the center of the escribed sphere touching the face II opposite the inscribed sphere; b, d, e, a', c', f' determine E_2, the center of the escribed sphere touching face III opposite the inscribed sphere; c, e, f, a', b', d' determine E_3, the center of the escribed sphere touching the face IV opposite the inscribed sphere.

Again a, e, b', c', d', f' determine a point E_4, equidistant from all four faces, the distances from I and II having the same sign; and the distances from III and IV having the other sign. Similarly b, f, a', c', d', e' determine E_5, whose distances from I and III have the same sign and its distances from II and IV the other sign. And c, d, a', b', e', f' determine E_6, whose distances from I and IV have the same sign and its distances from II and III the other sign.

This gives us seven escribed spheres in all.

In the case where the tetrahedron is regular, three of these ex-centers, the three last named, E_4, E_5, E_6, are at infinity. To prove this, consider fig. 2. Bisect CD at F. The plane of ΔABF is perpendicular to edge CD and $\angle AFB$ is the plane angle of the dihedral angle CD. In ΔABF, if $AB = l$, $AF = BF = l\sqrt{3}/2$. Hence $\sin F/2 = 1/\sqrt{3} = \sin \angle AFI$. Draw planes bisecting AC,

CD, DA externally (planes b′, c', e'). These meet at E_1, forming a tetrahedron $ACDE_1$. Consider the vertex A. Face angle $CAD = 60°$, dihedral angle $AC = \cos^{-1} 1/\sqrt{3}$, dihedral angle $AD = \cos^{-1} 1/\sqrt{3}$. By spherical trigonometry, law of cosines, we find dihedral angle $AE_1 = 90°$. Similarly, considering vertices C and D, we find dihedral angles CE_1, DE_1 to be 90°. Hence $AE_1 \perp E_1C$, E_1D; $AE_1 \perp$ plane of CE_1D; $AE_1 \perp E_1F$. But IF is $\perp E_1F$, whence AE_1 is $\parallel$ to IF.

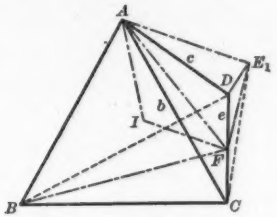

FIG 2.

Hence the line of intersection of a, b', and c' is parallel to the plane e and meets it only at infinity. So the point E_4 is at infinity. Similarly it may be shown that E_5 and E_6 are at infinity in this case.

This is the limiting case. In general AE_1 meets IF either on AE_1 produced or on E_1A produced.

Returning to fig. 1, A, E_1, E_4; A, E_2, E_5; A, E_3, E_6 are collinear. Also B, E_2, E_6; B, E_3, E_5; B, E_7, E_4; C, E_1, E_6; C, E_3, E_4; C, E_7, E_5; D, E_1, E_5; D, E_2, E_4; D, E_7, E_6.

Now if A, E_1, E_4 are collinear in that order (see fig. for Geom. 417), it is evident that the orders E_2, D, E_4; E_3, C, E_4; B, E_7, E_4; are determined. If A, E_2, E_5 are in that order, E_3, B, E_5; E_1, D, E_5; C, E_7, E_5 are determined orders. If A, E_3, E_6 are in that order, E_1, C, E_6; E_2, B, E_6; D, E_7, E_6 are determined orders.

If, however, E_1, A, E_4 be the order (see fig. 1), the orders D, E_2, E_4; C, E_3, E_4; E_7, B, E_4 obtain. And similarly with E_2, A, E_5 and E_3, A, E_6.

In special cases one, two, or three of the points E_4, E_5, E_6 may be at infinity. In that case the corresponding outer escribed sphere or spheres would not exist. We may therefore have four, five, six, or seven escribed spheres to a tetrahedron.

There are *two* types. (1) One of the vertices may be outside all three of the 'edgal' compartments in which the outer escribed spheres lie. In this case we may think of that vertex as at the top of the tetrahedron. The inscribed and three of the inner escribed spheres will then be above the level of the corresponding base, while the fourth inner and the three outer escribed spheres will be below that base. (2) One of the vertices may be at a corner of each of the three 'edgal' compartments in which the outer escribed spheres lie. In this case the three outer escribed spheres are above that vertex, the inscribed and three of the inner escribed spheres lie at a level between the vertex and its base and one of the inner escribed spheres only is below that base. It will be found

that if all seven escribed spheres are present the configuration can always be placed in one of these two types.

If one, two, or three of the outer spheres are lacking it will be found that we can place the configuration in *either* of the two types named above according as we consider the centers of the spheres which are at infinity to be *above* or *below* the vertex in question. These special cases occupy a limiting position between the two main types.

It may be pointed out that each vertex of the tetrahedron is collinear with four pairs of the eight points $I, E_1, E_2, E_3, E_7, E_4, E_5, E_6$. Fig. 1 shows the second type of tetrahedron; the figure that accompanies Geom. 417 shows the first type.

Editorial Note.—The discussion, scarcely intuitional, is printed for the division of the general case (7 finite ex-centers) into two sub-types. The intuitional view-point for the division into types with 4, 5, 6, 7 escribed spheres desired by the proposer is afforded by considering the angles at opposite pairs of edges. If equal the corresponding ex-center is at infinity, if unequal the ex-center is in the compartment of greater angle.

436. Proposed by A. J. KEMPNER, University of Illinois.

Given in a plane two similar curves arbitrarily situated, except that they shall possess the same sense of direction (which, of course, does not mean that they shall be similarly located). Let corresponding points on both curves be joined by straight lines, and let all of these straight lines be divided in the same ratio $\lambda : 1$, λ being any real number. Prove that the points of division all lie on a curve similar to the two given curves except when they all happen to fall together.

SOLUTION BY H. T. BIGELOW, Lafayette, Indiana.

Let the parametric equations of one of the curves be

$$x_1 = f(t), \qquad y_1 = \varphi(t). \tag{1}$$

The second curve, by reason of the similarity, is derivable from the first by an expansion from the origin, a rotation about the origin, and a translation. Its equation is, therefore,

$$\begin{aligned} x_2 &= a + k\cos\vartheta \cdot f(t) - k\sin\vartheta \cdot \varphi(t), \\ y_2 &= b + k\sin\vartheta \cdot f(t) + k\cos\vartheta \cdot \varphi(t), \end{aligned} \tag{2}$$

and corresponding points on the two curves (1) and (2) are given by equal values of t.

The corresponding point on the third curve is given by the equations

$$x_3 = \frac{x_1 + \lambda x_2}{1+\lambda}, \qquad y_3 = \frac{y_1 + \lambda y_2}{1+\lambda};$$

and

$$x_3 = \frac{\lambda a}{1+\lambda} + \frac{(1+\lambda k\cos\vartheta)}{1+\lambda} f(t) - \frac{\lambda k\sin\vartheta}{1+\lambda}\varphi(t),$$

$$y_3 = \frac{\lambda b}{1+\lambda} + \frac{\lambda k\sin\vartheta}{1+\lambda} f(t) + \frac{1+\lambda k\cos\lambda}{1+\lambda}\varphi(t).$$

If now we set $a' = \lambda a/(1+\lambda)$ and $b' = \lambda b/(1+\lambda)$ and choose k' and ϑ' so as to

make $k' \cos \vartheta' = (1 + \lambda k \cos \vartheta)/(1 + \lambda)$, $k' \sin \vartheta' = \lambda k \sin \vartheta/(1 + \lambda)$, the equations of the third curve become

$$\begin{aligned} x_3 &= a' + k' \cos \vartheta' \cdot f(t) - k' \sin \vartheta' \cdot \varphi(t), \\ y_3 &= b' + k' \sin \vartheta' \cdot f(t) + k' \cos \vartheta' \cdot \varphi(t). \end{aligned} \tag{3}$$

These equations are of the form (2) and consequently represent a curve similar to (1) and (2).

437. Proposed by J. BROOKS SMITH, Hampden Sidney, Va.

Let D, E, F be three arbitrary points taken on the sides of a triangle ABC. If Δ and Δ' be the areas of the triangles ABC and DEF, show that

$$\frac{\Delta'}{\Delta} = \frac{AF \cdot BD \cdot CE + AE \cdot CD \cdot BF}{abc},$$

the sign of each factor being determined as follows: Each segment adjacent to one of the vertices of the triangle ABC is to be regarded as positive or negative according as it is drawn towards or from the other vertex of the side containing the segment.

SOLUTION BY PAUL CAPRON, Annapolis, Md.

Let $AF = m$, $AE = q$, $BF = r$, $BD = k$, $CD = p$, $CE = l$; and let the areas of the triangles be $AFE = \Delta_1$, $BDF = \Delta_2$, $CED = \Delta_3$. Then, a, b, c being positive we have, in accordance with the given convention of signs, and the further convention that the area of a Δ shall be positive if the cyclic arrangement of its vertices is contra-clockwise, negative otherwise:

$$\frac{\Delta_1}{\Delta} = \frac{mq}{bc}, \qquad \frac{\Delta_2}{\Delta} = \frac{kr}{ca}, \qquad \frac{\Delta_3}{\Delta} = \frac{lp}{ab};$$

$$\frac{\Delta - \Delta_1 - \Delta_2 - \Delta_3}{\Delta} = \frac{\Delta'}{\Delta} = \frac{abc - amq - bkr - clp}{abc}.$$

Substituting $a = k + p$, $b = l + q$, $c = m + r$, we have

$$\frac{\Delta'}{\Delta} = \frac{klm + pqr}{abc} = \frac{AF \cdot BD \cdot CE + AE \cdot CD \cdot BF}{abc}.$$

Also solved by S. W. REAVES, H. C. FEEMSTER, C. N. SCHMALL, HORACE OLSON, and the PROPOSER.

438. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

By means of the theorem that the product of the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two products of pairs of opposite sides, obtain the usual formulas for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$ in terms of $\sin \alpha$, $\sin \beta$, $\cos \alpha$, $\cos \beta$.

SOLUTION BY A. M. HARDING, University of Arkansas.

Suppose PR is a diameter and $\angle SPR = \alpha$ and $\angle RPQ = \beta$. (Fig. 1.) Then

$$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$$

or

$$2r \cdot 2r \sin(\alpha + \beta) = 2r \cos \alpha \cdot 2r \sin \beta + 2r \cos \beta \cdot 2r \sin \alpha.$$

Hence,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Suppose PQ is a diameter and $\angle SPQ = \alpha$ and $\angle RPQ = \beta$. (Fig. 2.) Then

$$PR \cdot QS = PS \cdot RQ + PQ \cdot SR$$

or

$$2r\cos\beta \cdot 2r\sin\alpha = 2r\cos\alpha \cdot 2r\sin\beta + 2r \cdot 2r\sin(\alpha-\beta).$$

Hence,

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Suppose PQ is a diameter and $\angle RPQ = \alpha$ and $\angle SQP = \beta$. (Fig. 3.)

FIG. 1.

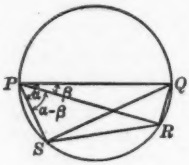

FIG. 2.

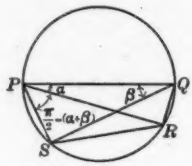

FIG. 3.

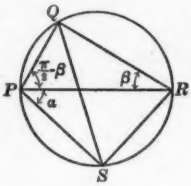

FIG. 4.

Then

$$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$$

or

$$2r\cos\alpha \cdot 2r\cos\beta = 2r\sin\beta \cdot 2r\sin\alpha + 2r \cdot 2r\sin[\pi/2 - (\alpha+\beta)].$$

Hence,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Suppose PR is a diameter and $\angle SPR = \alpha$ and $\angle PRQ = \beta$. (Fig. 4.) Then

$$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$$

or

$$2r \cdot 2r\sin[\pi/2 + (\alpha-\beta)] = 2r\cos\alpha \cdot 2r\cos\beta + 2r\sin\beta \cdot 2r\sin\alpha.$$

Hence,

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Also solved by C. N. SCHMALL.

439. Proposed by CHARLES N. SCHMALL, New York City.

Show that the areas of any two triangles circumscribed about the same circle are in the same ratio as their perimeters.

SOLUTION BY A. L. MCCARTY, Cape Girardeau, Mo.

Let the radius of the circle be r and the sides of the circumscribed triangles be a, b, c and d, e, f respectively.

Now it is evident that the area of the first triangle is $\frac{1}{2}r(a+b+c)$ and the area of the second is $\frac{1}{2}r(d+e+f)$. Hence, the two triangles are to each other as their perimeters.

Also solved by HORACE OLSON, J. L. RILEY, H. C. FEEMSTER, C. E. FLANAGAN, CLIFFORD N. MILLS, C. E. GITHENS, GEO. W. HARTWELL, ELMER SCHUYLER, WALTER C. EELLS, and A. M. HARDING.

MECHANICS.

288. Proposed by C. E. HORNE, Westminister College, Colorado.

Show that the tangential velocity of a projectile at any point of its path is equal to the velocity it would have acquired in falling, under the influence of gravitation alone, from the directrix to the point in question.

SOLUTION BY A. M. HARDING, University of Arkansas.

The equation of the path in parametric form is

$$x = u\cos\alpha \cdot t, \qquad y = u\sin\alpha \cdot t - \tfrac{1}{2}gt^2,$$

where u is the initial velocity and α is the angle of projection.

Then

$$dx = u\cos\alpha\, dt, \qquad dy = (u\sin\alpha - gt)dt,$$

$$ds^2 = dx^2 + dy^2 = (u^2 - 2ug\sin\alpha \cdot t + g^2t^2)dt^2 = (u^2 - 2gy)dt^2.$$

Now it can be easily shown that the distance from the directrix to the X-axis is given by $d = u^2/2g$. Hence $ds^2 = (2gd - 2gy)dt^2$.

Whence

$$\frac{ds}{dt} = \text{velocity} = \sqrt{2g(d-y)}.$$

Hence, the velocity at any point is equal to the velocity it would have acquired in falling from the directrix.

Also solved by ELIJAH SWIFT, CLIFFORD N. MILLS, HORACE OLSON, J. W. CLAWSON.

NUMBER THEORY.

207. Proposed by A. J. KEMPNER, University of Illinois.

There are 80 positive integers < 100 containing no figure 9 against 19 containing at least one figure 9. (For integers < 1000 the numbers are 728 and 271 respectively.) One might be led to believe that for every positive integer M the number N_1 of positive integers $< M$ containing no figure 9 is always greater than the number N_2 of positive integers $< M$ containing at least one figure 9.

To prove: $\lim (N_1/N_2) = 0$ for $M \doteq \infty$.

SOLUTION BY LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Let N_2 in every case represent the number of positive integers from 1 to M inclusive, which contain the figure 9 at least once; while N_1 represents the number of positive integers from 1 to M inclusive, which do not contain the figure 9. Then for

$M = 10$, $\quad N_2 = 1$,

$M = 10^2$, $\quad N_2 = 9 \times 1 + 10$, or $9 + 10$.

$M = 10^3$, $\quad N_2 = [9 \times 1 + 10]9 + 10^2$, or $9^2 + 9 \cdot 10 + 10^2$.

$M = 10^4$, $N_2 = \{[9 \times 1 + 10]9 + 10^2\}9 + 10^3$, or $9^3 + 9^2 \cdot 10 + 9 \cdot 10^2 + 10^3$.

$M = 10^5$, $N_2 = 9^4 + 9^3 \cdot 10 + 9^2 \cdot 10^2 + 9 \cdot 10^3 + 10^4$.

. . . . . .

$M = 10^t$, $N_2 = 9^{t-1} + 9^{t-2} \cdot 10 + 9^{t-3}10^2 + \cdots + 9 \cdot 10^{t-2} + 10^{t-1}$.

$M = 10^{t+1}$, $N_2 = 9^t + 9^{t-1} \cdot 10 + 9^{t-2}10^2 + \cdots + 9^2 \cdot 10^{t-2} + 9 \cdot 10^{t-1} + 10^t$.

In general, for $M = 10^t$, $N_2 = (10^t - 9^t)/(10 - 9) = 10^t - 9^t$; $N_1 = 9^t$.

$$\lim_{t=\infty} \frac{N_1}{N_2} = \lim_{t=\infty} \left[\frac{9^t}{10^t - 9^t} \right] = 0.$$

For the more general values of M such as $a_0 10^s + a_1 10^{s-1} + \cdots + a_{s-1}10 + a_s$, wherein none of the a's is equal to 9,

$$N_2 = a_0(10^s - 9^s) + a_1(10^{s-1} - 9^{s-1}) \cdots + a_{s-1}(10 - 9),$$

while

$$N_1 = a_0 9^s + a_1 9^{s-1} + \cdots + a_{s-1}9 + a_s.$$

Hence,

$$\lim_{s=\infty} \text{ of } \frac{N_1}{N_2} = 0.$$

If a_3 should be the first a which is 9, then

$$N_2 = a_0(10^s - 9^s) + a_1(10^{s-1} - 9^{s-1}) + a_2(10^{s-2} - 9^{s-2}) + a_3(10^{s-3} - 9^{s-3}) + a_4 10^{s-4} + \cdots + a_{s-1}10 + a_s + 1,$$

while

$$N_1 = a_0 9^s + a_1 9^{s-1} + a_2 9^{s-2} + a_3 9^{s-3} - 1.$$

Hence,

$$\lim_{s=\infty} \frac{N_1}{N_2} = 0.$$

With very slight changes the formula may be made to apply to the corresponding cases for 0, 1, 2, $\cdots$ 8.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

Editorial Note.—Several questions of interest are still outstanding. Question 4 is important for teaching analytic geometry (see December, 1913, issue). All efforts to increase the number of concrete applications available are to be commended.

Question 6 (February, 1914, issue) is of great significance in view of the present strong tendency toward increasing the vocational training in the schools.

Question 8 (March, 1914, issue) has an important bearing upon the relation of the colleges to the schools of education. Should not professional training for

teachers include practice teaching? In some cases only *observation* is expected, in others the practice teachers are put in actual charge of classes for a semester and given full responsibility under supervision. It has long been recognized that the training of teachers for elementary schools should include practice teaching; there is now at least one state college for the preparation of high school teachers where the responsible practice teaching is an important feature; why should not some sort of practice teaching be required of advanced students who are preparing for college positions?

Question 9 (March issue), so far as it pertains to coördinated courses in college mathematics, has just acquired a fresh interest in the new book by Professor Slichter, of the University of Wisconsin, entitled "Elementary Mathematical Analysis," which is designed to provide work equivalent to the traditional courses in trigonometry, college algebra, and analytic geometry. This book will be reviewed in the MONTHLY at an early date. Meanwhile, contributions concerning the general question of coördinated collegiate mathematics are desired.

Questions 11 to 15 (issues of May, June, September, and October) are of more limited interest, but nevertheless represent precisely the attitude of mind that this department is striving to foster, namely, a real question about a real difficulty encountered, or a genuine desire to gain further light upon important matters connected with the teacher's work.

Question 13 is possibly an exception to this last statement in that it seems to be an inquiry as to what is going on in the world outside rather than an expression of inner conflict of opinion. But, even so, it is a most worthy question. The teacher must look to the world outside for inspiration and courage to escape from submersion in the quiet waters of the daily routine. Professor Lehmer in California was surprised to find that Professor Bussey in Minnesota did not know about his course in synthetic projective geometry which he had been giving to junior college students for ten years. But how can we know about these things unless we *communicate* them to others? And how can we communicate them without a *medium* of communication? This is the mission of the MONTHLY and especially of the department of "Miscellaneous Questions." If every one who has something good which he has been concealing will come forward and let his light shine for the benefit of others, there will be a general awakening all along the line.

NEW QUESTIONS.

16. To what extent should a first course in geometry be made to appeal to a student's intuition? Should the subject be presented in a manner so as to depend to the greatest possible degree upon his previous experience, or is it desirable to attempt to make it more abstract and formal?

17. In analytic geometry, simplicity and directness are gained by making the condition for the collinearity of three points and the equation of the straight line depend upon the determinant formula for the area of a triangle. Similar advantages are gained by making the condition of complanarity of four points and the equation of the plane depend upon the determinant formula for the volume of a tetrahedron. The former is given in the texts. Why should not the latter be given? A uniform method of developing these two determinants is desired from some contributor.

18. In view of the present pressure for saving time and gaining efficiency, what are the most important sources of economy in the mathematical courses of the high school and the first two years in college?

NOTES AND NEWS.

EDITED BY W. DEW. CAIRNS.

The mathematical physical society of Kazan, Russia, offers the Lobatchewski prize of 500 roubles for productions relative to non-euclidean geometry, these to be submitted by Nov. 4, 1914.

Professor WILLIAM J. MILNE, Ph.D., LL.D., president of the New York State Normal College, Albany, N. Y., and author of many school texts in arithmetic, algebra and geometry, died September 4 at Bethlehem, N. H.

A second edition of Professor E. L. THORNDIKE's "Introduction to the Theory of Mental and Social Measurements" which was published in 1904 (Science Press, New York) has recently appeared.

In the May number of *Popular Science Monthly* occurs a paper by Professor WILLIAM MARSHALL on "The theory of relativity and the new mechanics."

Professor W. F. OSGOOD now holds the Perkins professorship of mathematics at Harvard University, the chair formerly held by Professor W. E. BYERLY.

Harvard University conferred the degree of doctor of science upon Professor W. C. SABINE at the commencement in June, 1914. He now holds the Hollis professorship of mathematics and natural philosophy at Harvard, the chair formerly held by Professor B. O. PEIRCE.

The thirty-fourth regular meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Monday and Tuesday, December 28, 29, 1914, the first session opening at 10 o'clock a. m., in Room 32, Ryerson Physical Laboratory.

Professor R. C. ARCHIBALD has an article in the May *Bulletin of the American Mathematical Society* on "Time as a fourth dimension."

A warm tribute by Professor JOHN TROWBRIDGE to the memory of the late Professor B. O. PEIRCE, of Harvard University, will be of great interest to the many American physicists and mathematicians who have studied with Professor Peirce. This appears in a recent number of the *Harvard Graduates' Magazine*.

Mr. R. A. JOHNSON, instructor in mathematics in Adelbert College, Western Reserve University, offers for the first semester of the year 1914–15 a teachers' course in elementary geometry. The meetings of the course are held at four o'clock in the afternoon, an hour which makes possible the attendance of teachers in the Cleveland public schools.

At the tercentenary celebration at the University of Groningen, the degree of doctor of science was conferred upon Professor E. B. VAN VLECK of the University of Wisconsin.

Professor F. ANDEREGG, of Oberlin College, has returned after a year's leave of absence. While in London he submitted to an operation for appendicitis, from which he recovered speedily.

A reprint from "*The Merchistonian*" sent to the editor by Professor F. CAJORI from Edinburgh contains a careful historical description of the Tower of Merchiston, later Merchiston House, and a brief but entertaining sketch of the life of John Napier with the reproduction of two of his portraits.

W. E. EDINGTON, B.A., Indiana State Normal School, graduate student in the University of Chicago, 1912–13, and instructor in the University of Colorado, 1913–14, has been elected professor of mathematics in the University of New Mexico at Albuquerque.

Because considerable difficulty has been experienced now and then by American teachers in obtaining the reports of the Mathematical Association, we reprint the following notice from the *Mathematical Gazette*, the organ of the Association:

The following Reports have been issued by the Association:—(i) "Reports on the Teaching of Elementary Mathematics," 1902–1908 (Geometry, Arithmetic and Algebra, Elementary Mechanics, Advanced School Mathematics, the Course required for Entrance Scholarships at the Universities, Mathematics in Preparatory Schools), price 6d. net; (ii) "Revised Report on the Teaching of Elementary Algebra and Numerical Trigonometry" (1911), price 3d. net; (iii) "Report on the Teaching of Mathematics in Preparatory Schools," price 3d. net; and (iv) "Report on the Correlation of Mathematical and Science Teaching," by a Joint Committee of the Mathematical Association and the Association of Public School Science Masters, price 6d. net; (v) A General Mathematical Syllabus for Non-Specialists in Public Schools, price 2d. net. These reports may be obtained from the Publishers of the *Gazette*, G. Bell and Sons, Ltd., Portugal St., Kingsway, London.

Among the articles contained in "The Mathematics Teacher" in the issues of March and June, 1914, the following may be mentioned as of interest to our readers: "Mathematics as a means to culture and discipline" (13 pages) by A. D. Yocum; "What mathematical knowledge and ability may reasonably be expected of the student entering college" (8 pages) by J. N. Hart; "Freshman algebra and the average freshman" (16 pages) by Ella D. Gray; and "Mathematics: the subject and the teacher" (12 pages) by F. C. Ferry.

The discussion of "A Kinetic Theory of Gravitation" by Charles F. Brush in *The Proceedings of the American Philosophical Society* for January–May, 1914, calls attention to a new way by which we may account for the immense forces in operation in the mutual attractions of astronomical bodies.

Today the science of mathematics is so vast and its development is so rapid that no one unaided could keep in touch with its growth in a single important field of considerable extent. Consequently it is necessary to have published reviews of the articles so that by means of these the individual worker will be able to determine quickly what are the recent additions to knowledge along the line in which he is especially interested. These reviews should meet two requirements which militate against each other, namely, they should be available promptly after the appearance of the article and they should be carefully prepared. The first of these requirements is met in *Revue Semestrielle des Publications Mathématiques*, an excellent and very prompt review of nearly all additions to mathematical literature; it does not appear to be used as much in this country as it might be. The second requirement is best met in the well-known *Jahrbuch über die Fortschritte der Mathematik*.

In recognition of the sixth international congress of mathematicians, to be held in Stockholm in 1916, King Gustave V has offered a gold medal bearing the likeness of Karl Weierstrass and a prize of 3,000 crowns, for any important discovery in the field of the theory of analytic functions. All manuscripts are to be submitted to *Acta mathematica* before October 31, 1915, the hundredth anniversary of the birth of Weierstrass.

The French Academy of Science offers the following prizes: (1) *Prix Bordin* for 1915 (3,000 francs), for making any noteworthy progress in the investigation of curves of constant torsion, and for determining if possible such of these curves as are algebraic, or at least such as are unicursal. (2) *Grand Prix des Sciences mathématiques* for 1916 (3,000 francs), for applying the methods of Henri Poincaré to the integration of linear algebraic differential equations, selected from the simplest. (3) *Prix Bordin* for 1917 (3,000 francs), for perfecting in any important way the arithmetical theory of non-quadratic forms. (4) *Prix Vaillant* for 1917 (4,000 francs), for determining and investigating all the surfaces which can be generated in two different ways by the displacement of an invariable curve.

Papers must be submitted to the Academy before the close of the year preceding the awarding of the various prizes.

Recent numbers of *Rendiconti del Circolo Matematico di Palermo* contain the following papers by Americans: Professor G. D. BIRKHOFF, "Note on the expansion problems of ordinary linear differential equations"; Professor J. EIESLAND, "On the algebraic curves of a tetrahedral complex and the surfaces conjugate to it"; Professors G. N. BAUER and H. L. SLOBIN, "Some transcendental curves and numbers"; Professor W. H. METZLER, "Rectangular arrays"; Dr. DUNHAM JACKSON, "A formula of trigonometric interpolation." This journal has published as a reprint the most important speeches and articles in memory of Henri Poincaré, together with a letter from Poincaré to the editor of *Rendiconti* on the subject of his last memoir, and a list of the memoirs of Poincaré which appeared in *Rendiconti* from 1888 to 1912.

The first regular meeting of the Association of Mathematics Teachers of New Jersey was held at Rutgers College, New Brunswick, N. J., on November 7th. Further notice of this meeting and of the purpose of this new organization will be given in a later issue of the MONTHLY.

The office of the American Mathematical Society, fitted out a year ago by Columbia University, was completely destroyed by a fire in University Hall on October 10, with loss of furniture, records, and stock of *Bulletin* and *Transactions* amounting to over $2000. The Society has duplicate stock of the *Transactions* and of the later years of the *Bulletin*. But the early volumes of the latter (as far as Vol. X) were completely lost. The Society will be glad to receive copies of these early volumes from its members and others who may be willing to donate them.

At the Pennsylvania State College, Dr. E. R. SMITH has been promoted to an associate professorship in mathematics and Dr. J. E. ROWE of Dartmouth College has been elected to an assistant professorship. It may be noted concerning the scope of work of this college that courses will be given during the fall term in theory of functions of real and complex variables, differential equations, functions defined by differential equations, statistical methods, Newtonian potential, and theory of numbers. Professor Stecker of this faculty gave one of the first courses in integral equations offered in the United States.

To pay proper tribute to the memory of Henri Poincaré and to link his name with an endowment for scientific purposes, a very representative international committee offers to the friends, colleagues and admirers of Poincaré the opportunity of sharing in a subscription which (1) shall furnish medals with the likeness of Poincaré, and (2) shall constitute a fund, the income from which is to be used by the Academy of Science to encourage younger scholars engaged in those branches of science whose development has been assured by his genius; namely, mathematical analysis, celestial mechanics, mathematical physics, philosophy of science. A bronze medal is given those subscribing from 25 to 50 francs, a silver medal for larger subscriptions. Subscriptions are to be sent to ERNEST LEBON, rue des Écoles, n° 4, Paris 5°.

The twenty-first summer meeting of the American Mathematical Society was held at Providence, R. I., on Tuesday and Wednesday, September 8, 9, 1914, as a part of the ceremonies connected with the celebration of the one-hundred and fiftieth anniversary of the founding of Brown University. There were fifty-two members present, including twelve from the middle west and far west, and thirty-three papers were presented during three half day sessions. The entertainment of the members given by President Faunce, Chancellor Chace, Professor Davis, and other members of the mathematical faculty of Brown University, was so generous and whole hearted that every one present will carry the most pleasant recollections of it for years to come. There was a luncheon,

an afternoon tea, a reception, a dinner, and last but not least a personally conducted tour about Newport under the guidance of Professors Richardson and Archibald, who as local members of the committee of arrangements were everywhere present and all things to all of us. President Faunce gave a notable address in his inimitable style at the dinner on Tuesday evening and Professor Carl Barus, of the department of physics in Brown University, gave an illuminating talk on "The relations of mathematics to physics." The mere enumeration of all these good things, not to mention the friendly discussions of the scientific papers presented and the general good fellowship always found on these occasions, should be sufficient to make every member of the Society resolve to attend the next meeting, which is to be held in San Francisco during the summer of 1915.

SPECIAL ANNOUNCEMENT.

As important evidence that the MONTHLY will maintain in 1915 the same high standard which has been set in the two years since its reorganization, we call attention to the following announcement:

Under the title " History of Zeno's Arguments on Motion," Professor Cajori will begin in the January, 1915, issue a series of articles which promise to be of unusual interest. Our present well-organized theory of limits, like every modern system of thought in all fields, has its roots far back in human history. Just as the ideas of negative and imaginary numbers struggled for recognition and formulation through hundreds of years, so the notion of a variable approaching a limit and all of the far-reaching consequences connected with the consideration of such a concept have gone through a process of slow development. Professor Cajori traces this development through a two-thousand-year struggle for light in which practically all of the great mathematicians and philosophers of the time were engaged. These articles will begin with the Volume XXII and will extend well through the year 1915. The editors are greatly pleased that we are able to offer such an attraction to the readers of the MONTHLY for the coming year. No teacher who has to deal with the theory of limits can afford to miss this latest product of Professor Cajori's historical researches.

As most subscriptions expire with the December issue, the renewal notices will be sent out in a few days, in order that all may have ample time to renew their subscriptions before the end of the year, and thus be sure of receiving the January issue containing the first instalment of Professor Cajori's articles.

The editors will greatly appreciate your prompt action in this matter, and they would suggest that you can do a real service to those in your acquaintance by calling their attention to this unique opportunity and sending their names to the Managing Editor for sample copies.

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THE
AMERICAN
MATHEMATICAL MONTHLY

VOLUME XXI DECEMBER, 1914 NUMBER 10

THE THEOREM OF ROTATION IN ELEMENTARY MECHANICS.[1]

By EDWARD V. HUNTINGTON, Harvard University.

In order to discuss the motion of a rigid body in a plane, we need to know two things: (1) the translational motion of some one point of the body; and (2) the angular motion of some one line of the body.

Information in regard to the first point is obtained from the well known theorem on the motion of the center of mass.

Information in regard to the second point is obtained from a theorem which is variously known as the theorem of rotation, the theorem of moments, the theorem of angular acceleration, etc.

This theorem of rotation is usually written in the form

$$T = I\dot{\omega},$$

where T = the torque, or turning moment, of the external forces about some suitably chosen point Q, I = the moment of inertia of the body about that point, and $\dot{\omega}$ = the angular acceleration of the body.

In order to bring out the "dimensions" of the quantities involved, the equation may be written in the more explicit form

$$\Sigma Fp = \frac{W}{g} k^2 \frac{d\omega}{dt},$$

where ΣFp = the sum of the moments of the external forces about the given point Q, k = the radius of gyration of the body about that point, and $d\omega/dt$ = the angular acceleration of the body at the instant in question.

This equation may be called the equation obtained by "taking moments about the point Q," and the question at once arises: for what choices of the point Q will the equation be valid? In other words, about what points is it legitimate to "take moments"?

Surprising as it may seem, the elementary text books are not in agreement in their answer to this question.

[1] Presented to the American Mathematical Society at the Summer meeting, September 8, 1914.